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Engineering Mathematics-4
(B.Tech,B.Sc,M.Sc)
(Part-1)

Unit-1

Partial Differential Equation

We consider partial differential equation of first degree (i.e. linear) in p and q.

$P_p + Q_q = R$ ………………………………………………………(1)

and the auxiliary equation is

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ……………………………………………………(2)

Now solving then we get two independent solutions u = a and v = b , where a , b are arbitrary constants . Then Ø (u ,v) = 0 is the general solution of the equation (1) .

Case-1 : Linear partial differential equation of first order

Q.No 1. Solve the following partial differential equations :

(i) yzp – xzq = xy (ii) y^2p – xzq = x (z – 2y) (iii) x^2p + y^2q = (x + y) z

Solution . (i) **yzp – xzq = xy** ………………………………(i)

we know that $P_p + Q_q = R$ ……………………………(2)

then P = yz , Q = - xz , R = xy

Now auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

or $\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$

Taking 1st & 2nd then

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

or $\frac{dx}{y} = \frac{dy}{-x}$ ⇒ ydy + xdx = 0

on integrating then

$\frac{y^2}{2} + \frac{x^2}{2} = c$

or $x^2 + y^2 = c_1$ ………………………………………(3)

Similarly, taking 1st & 3rd then

$\frac{dx}{yz} = \frac{dz}{xy}$ ⇒ $\frac{dx}{z} = \frac{dz}{x}$

or xdx – zdz = 0 on integrating then

$x^2 - z^2 = c_2$ ……………………………………(4)

Hence the general solution is $\emptyset\,(x^2 + y^2 , x^2 - z^2) = 0$

(ii) **$y^2p - xyq = x (z - 2y)$**

General equation is $P_p + Qq = R$

then $P = y^2$ $Q = - xy$ $R = x (z - 2y)$

Now, the Auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or $$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking 1st & 2nd then

$$\frac{dx}{y^2} = \frac{dy}{-x} \Rightarrow \frac{dx}{y} = \frac{dy}{-x} \quad \text{or} \quad ydy + xdx = 0$$

on integrating then

$$\frac{y^2}{2} + \frac{x^2}{2} = c \Rightarrow y^2 + x^2 = c_1$$

Similarly, we consider 2nd & 3rd then

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Now, solving then $y^2 - yz = c_2$

Hence the general solution is $\emptyset\,(y^2 + x^2 , y^2 - yz) = 0$

(iii) **$x^2p + y^2q = (x + y)z$**

Given equation is $x^2p + y^2q = (x + y)z$ …………………..(1)

we know that $P_p + Qq = R$ …………………………….(2)

From (1) and (2) then

$P = x^2$ $Q = y^2$ $R = (x + y)z$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

or $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$ …………………………………(3)

Taking 1^{st} & 2^{nd} then

$\frac{dx}{x^2} = \frac{dy}{y^2}$ on integrating then we get

$\frac{1}{x} - \frac{1}{y} = c_1$

Similarly from equation (3) then we have

$\frac{dx-dy}{x^2- y^2} = \frac{dz}{z(x+y)}$

Now solving then we have

$\frac{dx}{x-y} - \frac{dy}{x-y} = \frac{dz}{z}$ on integrating then

$\frac{x-y}{z} = c_2$

Hence, the general solution is $\emptyset\left(\frac{1}{x} - \frac{1}{y}, \frac{x-y}{z}\right) = 0$

Q No 2. Solve the following differential equation .

(i) $\frac{y^2z}{x}$ p + xzq = y^2

(ii) pz – qz = z^2+ $(x + y)^2$

(iii) $(x^2 – yz)p + (y^2 – zx)q = z^2 – xy$.

Solution .(i) $\frac{y^2z}{x}$ p + xzq = y^2 ……………………………….(1)

we know that $P_p + Q_q = R$ ……………………………….(2)

$p = y^2z$ $q = x^2z$ $R = y^2x$

Now we consider auxiliary equation

$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$ ……………………………….(3)

Taking 1^{st} & 2^{nd}

$\frac{dx}{y^2z} = \frac{dy}{x^2z}$ $\Rightarrow$ x^2 dx = y^2 dy on solving then

$x^3 – y^3 = c_1$

Similarly, taking 1^{st} & 3^{rd} then

$\frac{dx}{y^2z} = \frac{dz}{y^2x}$ $\Rightarrow$x dx = z dz on integrating then

$x^2 – z^2 = c_2$

Hence the general solution is $\emptyset$ $(x^3 – y^3 , x^2- z^2) = 0$

(ii) pz – qz = $z^2 + (x + y)^2$ …………………………..(1)

we know that $P_p + Q_q = R$ …………………………..(2)

then p = z q = -z $R = z^2 + (x + y)^2$

Now the auxiliary equation is

$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$

Taking 1st & 2nd then

dx = - dy $\Rightarrow$ dx + dy = 0

or $x + y = c_1$

Similarly , 1st & 3rd then

$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$ on solving then

In $(x^2 + y^2 + 2xy + z^2) - 2x = c_2$

Hence , the general solution is

$\emptyset$ {x + y , In $(x^2 + y^2 + 2xy + z^2) - 2x$ } = 0

(iii)Given $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$
……………………………….(1)

we know that $P_p + Q_q = R$ …………………………(2)

$p = x^2 - yz$ $q = y^2 - zx$ $R = z^2 - xy$

Now the auxiliary equation is

$\frac{dx}{x^2- yz} = \frac{dy}{y^2- zx} = \frac{dz}{z^2- xy}$

or

$\frac{dx-dy}{x-y} = \frac{dy-d}{y-z} = \frac{dz-d}{z-x}$

Taking 1st & 2nd then we get

In (x – y) = In (y – z) + In c

or $\frac{x-y}{y-z} = c_1$

Similarly, for the second solution we have

$$\frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xy} = \frac{x\,dx+y\,dy+z\,dz}{(x+y+z)\{x^2+y^2+z^2-xy-yz-zx\}} \quad \text{................(3)}$$

and $$\frac{dx+dy+dz}{x^2+y^2+z^2-yz-zx-xy} \quad \text{.....................................(4)}$$

Now from equation (3) & (4) we have

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x+y+z} = dx + dy + dz$$

$$x\,dx + y\,dy + z\,dz = (x + y + z)(dx + dy + dz)$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} - c_2$$

solving we get $xy + yz + zx = c_2$

Hence , the general solution is

$$\emptyset\left\{\frac{x-y}{y-z}, xy + yz + zx\right\} = 0$$

Q.No 3 . Solve the P.D.E

(i) $(mz - ny)p + (nx - lz)q = ly - mx$

(ii)$x^2(y - z)p + y^2(z - x)q = z^2(x - y)$

Solution. (i) $(mz - ny)p + (nx - lz)q = ly - mx$ ……………………………..(1)

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \quad \text{……………………………….(2)}$$

multiplying x , y , z in each then

$$\frac{x\,dx}{xmz-xny} = \frac{y\,dy}{ynx-ylz} = \frac{z\,dz}{zly-zmx}$$

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{xmz-xny+ynx-ylz+zly-zmx} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

or $x\,dx + y\,dy + z\,dz = 0$ on integrating then

$x^2 + y^2 + z^2 = c_1$

Similarly, multiplying l , m , n in each then

$$\frac{l\,dx+m\,dy+n\,dz}{lmz-lny+mnx-mlz+nly-m} = \frac{l\,dx+mdy+ndz}{0}$$

or $l\,dx + m\,dy + n\,dz = 0$ on integrating we get

$lx + my + nz = c_2$

Hence the general solution is

$\emptyset\ \{x^2 + y^2 + z^2,\ lx + my + nz\} = 0$

(ii)Given $x^2 (y - z)p + y^2 (z - x)q = z^2 (x - y)$ ………………………..(1)

we know that $P_p + Q_q = R$ ………………………………….(2)

The auxiliary equation is

$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$ ……………………………………(3)

case 1st : from equation (3) we write

$\frac{dx/x^2}{y-z} = \frac{dy/y^2}{z-x} = \frac{dz/z^2}{x-y}$

or $\frac{1}{x^2}$ dx + $\frac{1}{y^2}$ dy + $\frac{1}{z^2}$ dz = 0 on integrating then we get

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$

Similarly, we consider

case 2nd : from equation (3)we have

$\frac{dx/x}{x(y-z)} = \frac{dy/y}{y(z-x)} = \frac{dz/z}{z(x-y)}$

or $\frac{1}{x}$ dx + $\frac{1}{y}$ dy + $\frac{1}{z}$ dz = 0 on integrating then

ln xyz = ln c_2 or xyz = c_2

Hence the general solution is

$\emptyset \{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} ,\ xyz \} = 0$

Q.No 4. Solve the P.D. E

$x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$, where $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$.

Solution. Given

$x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$ ....(1)

we know that $P_P + Q_q = R$ ....(2)

the auxiliary equation is

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad \text{.......................................(3)}$$

case 1st : multiplying x , y , -1 in (3) then we have

$$\frac{x\,dx+y\,dy+z\,dz}{x^2y^2+x^2z-y^2x^2-y^2z-zx^2+zy^2} = 0$$

or $x\,dx + y\,dy - dz = 0$ on integrating then

$$\frac{x^2}{2} + \frac{y^2}{2} - z = c_1$$

or $x^2 + y^2 - 2z = c_1$

case 2nd : from equation (3) we write

$$\frac{dx/x}{y^2+z} = \frac{dy/y}{-(x^2+z)} = \frac{dz/z}{x^2-y^2}$$

or $\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2} = 0$ i.e., $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

on integrating then we write

$\ln xyz = \ln c_2$ or $xyz = c_2$

Hence the general solution is $\emptyset\{ x^2+y^2-2z ,\ xyz\} = 0$

Q.NO 5: Solve the P.D.E

$x (y^2-z^2)p+y\,(z^2-x^2)q=z\,(x^2-y^2)$

where, $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$

Solution. Given

$x\,(y^2 - z^2)p + y\,(z^2 - x^2)q = z\,(x^2 - y^2)$ ……………………………(1)

we know that $P_p + Q_q = R$ ……………………………(2)

Now the auxiliary equation is

$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y\,(z^2-x^2)} = \frac{dz}{z\,(x^2-y^2)}$ ………………………………(3)

case 1st : multiplying x , y , z in equation (3)then we get

$\frac{x\,dx+y\,dy+z\,dz}{x^2y^2-x^2z^2+y^2z^2-y^2x^2+z^2x^2-z^2y^2} = 0$

i.e., $x\,dx + y\,dy + z\,dz = 0$ on integrating then

$x^2 + y^2 + z^2 = c_1$

case 2nd : from equation (3) we have

$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2} = 0$ i.e., $\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

or $\ln\ xyz = \ln\ c_2$ then

$xyz = c_2$

Hence, the general solution is

$\emptyset\ \{x^2 + y^2 + z^2,\ xyz\} = 0$

QNO.6 : Solve the P .D .E

(i) p + 3q = 5 z + tan (y – 3x)

(ii)$(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$

(iii)$(y + zx)p - (x + yz)q = x^2 - y^2$

Solution. (i) Given

p +3q = 5z + tan (y – 3x) ……………………….(1)

we know that $P_p + Q_q = R$ ……………………….(2)

the auxiliary equation is

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)} \quad \text{……………………….(3)}$$

case 1st : from 1st & 2nd we write

$$\frac{dx}{1} = \frac{dy}{3} \Rightarrow 3\,dx = dy$$

on integrating then $y - 3x = c_1$

case 2nd : from 1st & 3rd we write

$$\frac{dx}{1} = \frac{dz}{5z+\text{ta}\ \ (y-3x)}$$

$dx = \frac{dz}{5z+\text{ta}\ \ (y-3x)}$ on integrating then

In { 5z + tan (y – 3x)} – 5x = c_2

Hence, the general solution is

Ø [(y – 3x) , In { 5z + tan (y – 3x)} – 5x] = 0

(ii)Given

$(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$ ……………………….(1)

we know that $P_p + Q_q = R$ ………………………….(2)

the auxiliary equation is

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} \quad \text{……………………………..(3)}$$

case 1st : from 2nd & 3rd we write

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \ln y = \ln z + \ln c_1$$

or $\frac{y}{z} = c_1$

case 2nd : multiplying x , y , z in equation (3) then we write

$$\frac{x\,dx+y\,dy+z\,dz}{-x\,(x^2+y^2+z^2)} = \frac{dz}{-2xz}$$

or $\frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2} = \frac{dz}{z}$

on integrating then $\ln(x^2 + y^2 + z^2) = \ln z + \ln c_2$

$$\frac{x^2+y^2+z^2}{z} = c_2$$

Hence, the general solution is

$$\emptyset \left\{ \frac{y}{z} , \frac{x^2+y^2+z^2}{z} \right\} = 0$$

(iii)Given that

$(y + zx)p – (x + yz)q = x^2 – y^2$ ....(1)

we know that $P_p + Q_q = R$ ....(2)

the auxiliary equation is

$$\frac{dx}{y+zx} = \frac{dy}{-(\,x+yz)} = \frac{dz}{x^2-y^2} \quad ....(3)$$

case 1st : multiplying x , y , - z then we write

$$\frac{x\,dx+y\,dy-z\,dz}{xy+zx^2-xy-y^2z-x^2z+zy^2} = 0$$

i.e., $x\,dx + y\,dy + z\,dz = 0$

on integrating then we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c_1$$

or $x^2 + y^2 – z^2 = c_1$

caser 2nd : multiplying y , x , 1 then we write

$$\frac{y\,dx+x\,dy+dz}{y^2+xyz-x^2-xyz+x^2-y^2} = 0$$

i.e., $y\,dx + x\,dy + dz = 0$ solving we get

$xy + z = c_2$

Hence, the general solution is

$\emptyset \{ x^2 + y^2 – z^2 ,\; xy + z \} = 0$

QNO 7. Solve the P. D . E

(i) $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$

(ii) $\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$

(iii) $(y^2 + z^2)p - xyq = -zx$

Solution . (i) Given that

$(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$ ……………………………(1)

we know that $P_p + Q_q = R$ ………………………………(2)

the auxiliary equation is

$$\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y^2-zx} = \frac{dz}{z\,(x-y)} \quad \text{……………………………(3)}$$

case 1st : multiplying 1 , -1 , -1 in equation (3) then we have

$$\frac{dx-dy-dz}{x^2-y^2-yz-x^2+y^2+zx-zx+zy} = 0 \quad \text{i.e.,} \quad dx - dy - dz = 0$$

on integrating then $x - y - z = c_1$

case 2nd : from equation (3) wehave

$$\frac{x\,dx-y\,dy}{x^3-y^2x-xyz-x^2y+y^3+xyz} = \frac{dz}{z(x-y)}$$

or
$$\frac{x\,dx-y\,dy}{x^3-xy^2-x^2y+y^3} = \frac{dz}{z(x-y)}$$

$$\frac{x\,dx-y\,dy}{(x-y)(x^2-y^2)} = \frac{dz}{z(x-y)} \Rightarrow \frac{x\,dx-y\,dy}{x^2-y^2} = \frac{dz}{z}$$

on integrating we have

$\text{In}(x^2 - y^2) = 2\,\text{In}\, z + 2\,\text{In}\, c_2$

or $\frac{x^2-y^2}{z^2} = c_2$

Hence the general solution is

$\emptyset\{x-y-z, \frac{x^2-y^2}{z^2}\} = 0$

(ii)Given that

$\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$ ....(1)

we know that $P_p + Q_q = R$ ....(2)

the auxiliary equation is

$$\frac{yz\,dx}{y-z} = \frac{zx\,dy}{z-x} = \frac{xy\,dz}{x-y}$$ ....(3)

or $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

or $\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y} = 0$ i.e., $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

on integrating we have

$\ln xyz = \ln c_1$ or $xyz = c_1$

case 2nd : from equation (3) we have

$$\frac{dx+dy+dz}{xy-xz+yz-yx+zx-zy} = 0$$

or $dx + dy + dz = 0$

on integrating we get $x + y + z = c_2$

Hence , the general solution is

$\emptyset$ { xyz , x + y + z } = 0

(iii)Given that

$(y^2 + z^2)p - xyq = -zx$ ....(1)

we know that $P_p + Q_q = R$ ....(2)

the auxiliary equation is

$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$ ....(3)

case 1st : from 2nd & 3rd we have

$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \frac{y}{z} = c_1$

case 2nd : from equation (3) we have

$\frac{x\,dx + \quad dy + z\,dz}{xy^2 + xz^2 - xy^2 - z^2x} = 0$ i.e., x dx + y dy + z dz = 0

on integrating we get

$x^2 + y^2 + z^2 = c_2$

Hence , the general solution is

$\emptyset\{\frac{y}{z},\ x^2 + y^2 + z^2\} = 0$

Case 2nd : Non- Linear Partial Differential Equation of First Order

(i)Equation of the form f (p , q) = 0 i.e., equations involving only p and q then complete solution is z = ax + by + c , where a and b are connected by the relation f(a , b) = 0 since p = $\frac{\partial z}{\partial x}$ = a & q = $\frac{\partial z}{\partial y}$ = b

Now, we find b in terms of a then let b = ∅ (a) therefore complete solution z = ax + ∅(a) y + c , where a and b are arbitrary constant.

QNO. 1: Solve the P.D.E

(i) $\sqrt{p} + \sqrt{q}$ = 1 (ii) p q = p + q

Solution . (i) The equation of the form f (p , q) = 0 ……………………(1)

and the complete solution is z = ax + by + c , where $\sqrt{a} + \sqrt{b}$ = 1

or b = $(1-\sqrt{a})^2$. Now from equation (1) the complete solution is

z = ax + $(1-\sqrt{a})^2$y + c . Ans.

(ii) The equation of the form is f (p , q) = 0

and the complete solution is z = ax + by + c , where ab = a + b

or b = $\frac{a}{a-1}$. Therefore complete solution is

z = ax + $\frac{a}{a-1}$ y + c . Ans.

(ii)Equation of the form z = px + qy + f (p , q) . Then the complete solution is z = ax + by + f (a , b) . where p = a & q = b

QNO.1: Solve the P . D . E

z = px + qy + $\sqrt{1 + p^2 + q^2}$

Solution . The given equation of the form is z = px + qy + f (p , q)

then its complete solution is z = ax + by + $\sqrt{1 + a^2 + b^2}$

Linear Partial Differential Equation With Constant Coefficient. we consider general equation i.e.,

$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2} \partial y} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}} + C_0 \frac{\partial z}{\partial x} + c_1 \frac{\partial z}{\partial y} + P_0 z = F(x, y)$ ……………………………………………(1)

where the coefficient A_0 , A_1 …………….. A_n ; B_0 , B_1 , ………………B_{n-1} ; c_0 , c_1 ……………………are constant or functions of x and y .