Engineering Mathematics-4
(B.Tech,B.Sc,M.Sc)
(Part-2)

Homogeneous Linear Partial Differential Equation with Constant Coefficients.

We consider

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\,\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\,\partial y^2} + \ldots\ldots\ldots\ldots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

………………………………….(1)

Where , a_0 , a_1 , a_2 ……………..a_n are constant is called a homogeneous linear partial differential equation of the nth order with constant coefficients . Now when D = $\frac{\partial}{\partial x}$ and D'= $\frac{\partial}{\partial y}$ then we write

$\emptyset$ (D , D') z = F (x , y) ………………………………….(2)

case 1st : C. F . i.e., complementary function which is the complete solution of the equation $\emptyset$ (D , D') z = 0.

Case 2nd : Particular integral (p.I) which is a particular solution of

$\emptyset$ (D , D') z = F (x , y)

The complete solution Z = C.F + P.I

QNO. 1 : Solve the P. D. E

(i) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$ **(ii)** $(D + 2D')(D - 3D')^2 z = 0$

(iii) $4r - 12s + 9t = 0$

Solution. (i) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

or $(D^2 - DD' - 6D'^2)\, z = 0$ where, $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$

The auxiliary equation is

$$m^2 - m - 6 = 0$$

$$m = -2, 3$$

$$C.F = f_1(y + 3x) + f_2(y - 2x)$$

and $P.I = 0$

Hence the complete solution $z = C.F + P.I$

or $z = f_1(y + 3x) + f_2(y - 2x)$ Ans.

(ii) The given P.D.E . is

$(D + 2D')(D - 3D')^2 z = 0$

The auxiliary equation is $(m + 2)(m - 3)^2 = 0$

$m = -2 , 3 , 3$

C.F $= f_1(y - 2x) + f_2(y + 3x) + x f_3(y + 3x)$

and P.I = 0

Hence the complete solution z = C.F + P.I

or $z = f_1(y - 2x) + f_2(y + 3x) + x f(y + 3x)$ Ans.

(iii) $4r - 12s + 9t = 0$

or $4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$

$(4m^2 - 12DD' + 9D'^2)z = 0$

The auxiliary equation is $4m^2 - 12m + 9 = 0$

$(2m - 3)^2 = 0$

m = 3/2 , 3/2

C.F = $f_1 (y + \frac{3}{2} x) + x f_2 (y + \frac{3}{2})$

and P.I = 0

Hence the complete solution z = C.F + P. I

or $z = f_1(y + \frac{3}{2}x) + x f_2(y + \frac{3}{2}x)$ **Ans.**

QNO.2 : Solve the P.D.E.

(i) $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial y^2} = 0$ **(ii)** $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

Solution. (i) The given equation is

$(D^3 - 3D^2D' + 2DD'^2)\, z = 0$

and the auxiliary equation is $m^3 - 3m^2 + 2m = 0$

$m = 0, 1, 2$

C.F $= f_1(y) + f_2(y + x) + f_3(y + 2x)$

and P.I $= 0$

Hence the complete solution $z =$ C.F + P.I

or $z = f_1(y) + f_2(y + x) + f_3(y + 2x)$ **Ans.**

(ii) The given equation is

$(D^4 - D'^4)\, z = 0$

The auxiliary equation is $m^4 - 1 = 0$

$(m^2 - 1)(m^2 + 1) = 0$

$m = \pm 1, \ \pm i$

$C.F = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$

and $P.I = 0$

Hence the complete solution $z = C.F + P.I$

or $z = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$ **Ans.**

QNO. 3 : Solve the P.D.E

$\frac{\partial^3 u}{\partial x^3}$ - 3 $\frac{\partial^3 u}{\partial x^2 \partial y}$ + 4$\frac{\partial^3 u}{\partial y^3}$ = e^{x+2y}

Solution. The given equation is

$(D^3 - 3D^2D' + 4D'^3)\, u = e^{x+2y}$ where $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$

The auxiliary equation is $m^3 - 3m^2 + 4 = 0$

m = 2 , 2 , -1

C.F = $f_1(y - x) + f_2(y + 2x) + x\, f_3(y + 2x)$

P.I = $\frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$

$= \frac{1}{27} \iiint e^u du\, du\, du$ where u = x + 2y $= \frac{1}{27} e^{x+2y}$

Hence the complete solution Z = C.F + P.I

or $Z = f_1(y - x) + f_2(y + 2x) + x\, f_3(y + 2x) + \frac{1}{27} e^{x+2y}$

where f_1, f_2 and f_3 are arbitrary functions.

QNO. 4 : Solve the P.D.E

$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

Solution. The given equation is

$(D^2 + 3\,DD' + 2\,D'^2)\,z = x + y$

Now the auxiliary equation is $m^2 + 3m + 2 = 0$

$m = -1\,,\,-2$

$C.F = f_1(y - x) + f_2\,(y - 2x)$

and P.I = $\frac{1}{D^2+3DD'+2D'^2}(x + y)$

$= \frac{1}{6}\iint u\,du\,du$ where $u = x + y$

$= \frac{1}{36}(x + y)^3$

Hence the complete solution Z = C.F + P.I

or $Z = f_1(y - x) + f_2(y - 2x) + \frac{1}{36}(x + y)^3$ **Ans.**

QNO.5 : Solve the P.D.E

$\frac{\partial^2 z}{\partial x^2}$ - 2 $\frac{\partial^2 z}{\partial x \partial y}$ + $\frac{\partial^2 z}{\partial y^2}$ = sin (2x + 3y)

Solution. The given equation is

$$(D^2 - 2\,DD' + D'^2)z = \sin(2x + 3y)$$

The Auxiliary equation is

$$m^2 - 2m + 1 = 0$$

m = 1 , 1

C.F = $f_1(y + x) + x\, f_2(y + x)$

Similarly, P.I = $\frac{1}{D^2-2DD'+D'^2}$ Sin (2x+3y)

$= \frac{1}{(D-D')^2}$ sin (2x + 3y)

$= \frac{1}{(2-3)^2} \iint u\, du\, du$ where, u = 2x+3y

$= \int (-\cos u)\, du$ = - sin u

$= - \sin(2x + 3y)$

Hence, the complete solution Z = C.F + P.I

$Z = f_1(y + x) + x f_2(x + y) - \sin(2x + 3y$

QNO.6: Solve the P.D.E

$(2D^2 - 5DD' + 2D'^2)z = 24(y - x).$

Solution. The Auxiliary equation is

$$2m^2 - 5m + 2 = 0$$

$$m = \frac{1}{2},\ 2$$

$C.F = f_1(y + \frac{1}{2}x) + f_2(y + 2x)$

$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 24(y - x)$

$= \frac{24}{9} \iint u\, du\, du$ where $u = y - x$

$= \frac{4}{9}(y - x)^3$

Hence the complete solution $Z = C.F + P.I$

or $Z = f_1(y + \frac{1}{2}x) + f_2(y + 2x) + \frac{4}{9}(y - x)^3$ **Ans.**

QNO.7 : Solve the P.D.E

r + s – 2t = $\sqrt{2x + y}$.

Solution : $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2}$ = (2x + y) $^{1/2}$

$(D^2 + DD' - 2D'^2)$ z = (2x + y)$^{1/2}$

The Auxiliary equation is

$$m^2 + m - 2 = 0$$

$$m = 1 , -2$$

C.F = f_1 (y + x) + f_2 (y - 2x)

P.I = $\frac{1}{D^2+DD'-2D'^2}$ (2 x + y)$^{1/2}$

= $\frac{1}{4}\iint u^{1/2} du\, du$ where u = 2x + y

= $\frac{1}{15}(2x + y)^{5/2}$

Hence complete solution Z = C.F + P .I or $Z = f_1(y + x) + f_2(y - 2x) + \frac{1}{15}(2x + y)^{5/2}$ **Ans.**

QNO.8 : Solve the P.D.E

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$$

Solution. The given equation is

$(D^2 - 2DD' + D'^2)\, z = \sin x$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$m = 1, 1$

$C.F = f_1(y + x) + x\, f_2(y + x)$

$$P.I = \frac{1}{D^2 - 2DD' + D'^2} \sin x$$

$$= \frac{1}{(D - D')^2} \sin(x + 0.y)$$

$$= \iint \sin u \, du \, du$$

= - sin (x + 0.y) = - sin x

Hence the complete solution Z = C.F + P.I

or Z = f_1(y + x) + x f_2(y + x) – sin x **Ans.**

QNO. 9 : Solve the P.D.E

$$2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 5 \sin(2x + y)$$

Solution. The given equation is

$(2D^2 - 5DD' + 2D'^2)\, z = 5 \sin(2x + y)$

The auxiliary equation is $2m^2 - 5m + 2 = 0$

$m = \frac{1}{2}, 2$

C.F = $f_1(y + \frac{1}{2}x) + f_2(y + 2x)$

P.I = $\frac{1}{2D^2 - 5DD' + 2D'^2}\, 5 \sin(2x + y)$

$= \frac{5x}{3}\int \sin u\, du = -\frac{5x}{3}\cos(2x + y)$

Hence the complete solution Z = C.F + P.I

or $Z = f_1(y + \frac{1}{2}x) + f_2(y + 2x) - \frac{5x}{3}\cos(2x + 5)$ **Ans.**

QNO. 10 : Solve the P.D.E

$(D^2 + 5DD' + 6D'^2)\, z = \frac{1}{y-2x}$

Solution. The auxiliary equation is

$m^2 + 5m + 6 = 0$

$m = -2, -3$

$C.F = f_1(y - 2x) + f_2(y - 3x)$

and $P.I = \frac{1}{D^2+5DD'+6D'^2}\left\{\frac{1}{y-2x}\right\}$

$= x\int \frac{1}{u}\,du = x \ln(y - 2x)$

Hence the complete solution Z = C.F + P.I

or $Z = f_1(y - 2x) + f_2(y - 3x) + x \ln(y - 2x)$ **Ans.**

QNO. 11 : Solve the P.D.E

$\frac{\partial^2 z}{\partial x^2}$ - 3 $\frac{\partial^2 z}{\partial x \partial y}$ + 2 $\frac{\partial^2 z}{\partial y^2}$ = e^{2x+3y} + sin (x - 2y)

Solution. The auxiliary equation is

$m^2 - 3m + 2 = 0$

m = 1 , 2

C.F = $f_1(y + x) + f_2(y + 2x)$

P.I = $\frac{1}{D^2-3DD'+2D'^2} e^{2x+3y} + \frac{1}{D^2-3DD'+2D'^2}$ sin (x - 2y)

= $\frac{1}{4} \iint e^u du\, du + \frac{1}{15} \iint \sin u\, du\, du$

= $\frac{1}{4} e^{2x+3y} - \frac{1}{15}$ sin (x – 2y)

Hence the complete solution Z = C.F + P.I

or Z = $f_1(y + x) + f_2(y + 2x) + \frac{1}{4} e^{2x+3y} - \frac{1}{15}$ sin (x – 2y) **Ans.**

QNO. 12 : Solve the P.D.E

$\frac{\partial^3 z}{\partial x^3}$ - 7 $\frac{\partial^3 z}{\partial x \partial y}$ - 6 $\frac{\partial^3 z}{\partial y^3}$ = sin (x + 2y) + e^{3x+y}

Solution. The auxiliary equation is

$m^3 - 7m - 6 = 0$

m = -1 , -2, 3

C.F = $f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$

P.I = $\frac{1}{D^3 - 7DD' - 6D'^3}$ sin (x + 2y) + $\frac{1}{D^3 - 7DD' - 6D'^3} e^{3x+y}$

= - $\frac{1}{75} \iiint \sin u \, du \, du \, du + \frac{x}{3D^2 - 7} \iint e^{3x+y} \, du \, du$

= - $\frac{1}{75}$ cos (x + 2y) + $\frac{x}{20} e^{3x+y}$

Hence the complete solution Z = C.F + P. I

or $Z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) - \frac{1}{75}\cos(x + 2y) + \frac{x}{20} e^{3x+y}$

Ans.